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Conjectures on index and algebraic connectivity of graphs

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ABSTRACT

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of a graph G is denoted by $A(G)$ and defined as the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ for $v_iv_j \in E(G)$ and 0 otherwise. The largest eigenvalue (λ_1) of $A(G)$ is called the spectral radius or the index of G. The Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of its vertex degrees and $A(G)$ is the adjacency matrix. Among all eigenvalues of the Laplacian matrix of a graph, the most studied is the second smallest, called the algebraic connectivity (a) of a graph [12]. In [1,2], Aouchiche et al. have given a series of conjectures on index (λ_1) and algebraic connectivity (a) of G (see also [3]). Here we prove two conjectures and disprove one by a counter example.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Also let d_i be the degree of vertex v_i for $i = 1, 2, \ldots, n$. The minimum vertex degree is denoted by δ and the maximum by Δ. Denote by $i \sim j$, vertices v_i and v_j are adjacent. For two vertices v_i and v_j, let $d(i, j)$ be the distance between v_i and v_j which is the number of edges in a shortest path joining v_i and v_j. The diameter d of a graph is the maximum distance between any two vertices of G. A pendant vertex of G is a vertex of degree 1. A pendant neighbor is a vertex adjacent to a pendant vertex. We suppose G has $q(G)$ pendant neighbors.

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The adjacency matrix of a graph G is denoted by $A(G)$ and defined as the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ for $v_iv_j \in E(G)$ and 0 otherwise. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The largest eigenvalue (λ_1) is called the spectral radius or the index of G. When more than one graph is under discussion, we may write $\lambda_i(G)$ instead of λ_i. The matrix $A(G)$ is well studied by several authors [4–6,8].

Let $D(G)$ be the diagonal matrix of order n whose diagonal entry d_i is the degree of the vertex v_i of graph G. The Laplacian matrix associated with a graph G are the $n \times n$ matrix given by $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of order n of graph G and $A(G)$ is the $(0, 1)$-adjacency matrix of graph G. Clearly, $L(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n = 0$. They are usually called the Laplacian eigenvalues of G. Among all eigenvalues of the Laplacian of a graph, the most studied is the second smallest, called the algebraic connectivity (a) of a graph. It is well known that a graph is connected if and only if $a = \mu_{n-1} > 0$. When more than one graph is under discussion, we may write $\mu_i(G)$ instead of μ_i. The matrix $L(G)$ is well studied by several authors [6,7,9–11,14–17,19–22].

A kite $Ki_{n,\omega}$ is the graph obtained from a complete graph K_ω and a path $P_{n-\omega}$ by adding an edge between a vertex from the complete graph and an end point from the path. The girth $g = g(G)$ of a graph G on n vertices with at least n edges is the length of the smallest cycle in G. A matching in a graph is a set of disjoint edges, and the maximum cardinality of a matching over all possible matching in a graph G is the matching number of G and denoted by $\mu = \mu(G)$. The vertex connectivity ν is the minimum number of vertices whose deletion will disconnect G or leave a single vertex, and the edge connectivity κ is the minimum number of edges whose deletion will disconnect G. As usual, K_n, $K_{1,n-1}$ and P_n denote respectively the complete graph, the star and the path on n vertices. In [1,2], Aouchiche et al. gave the following conjectures involving index, algebraic connectivity, girth, vertex connectivity, edge connectivity and matching number of G (see also [3]).

Conjecture 1 ([1,2,3])**.** Let G be a connected graph on n vertices with girth g and algebraic connectivity a. Then $a + g$ and $a \cdot g$ are minimum for the kite $Ki_{n,3}$.

Conjecture 2 ([1,2,3])**.** Let G be a connected graph on $n \geqslant 3$ vertices with index λ_1, vertex connectivity ν and edge connectivity κ. Then

$$\begin{aligned} &\lambda_1 - \nu \leqslant n - 3 + t; \quad \lambda_1 - \kappa \leqslant n - 3 + t; \\ &\frac{\lambda_1}{\nu} \leqslant n - 2 + t; \quad \text{and} \quad \frac{\lambda_1}{\kappa} \leqslant n - 2 + t; \end{aligned} \tag{1}$$

where t is such that $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2 - 3n + 1)t - 1 = 0$. Moreover, the equality holding in (1) if and only if $G \cong Ki_{n,n-1}$.

Conjecture 3 ([1,2,3])**.** Let G be a connected graph on n vertices with matching number μ and algebraic connectivity a. Then $a \cdot \mu \geqslant 1$ with equality holding if and only if $G \cong K_{1,n-1}$.

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we prove two conjectures and disprove one by a counter example.

2. Preliminaries

In this section, we shall list some previously known results that will be needed in the next section.

Lemma 2.1 (Perron–Frobenius [13])**.** *A non-negative matrix B always has a non-negative characteristic value r such that the modulii of all the characteristic values of B do not exceed r. To this 'maximal' characteristic value r there corresponds a non-negative characteristic vector* $\mathbf{Y}$ *such that*

$$B\mathbf{Y} = r\mathbf{Y}(\mathbf{Y} \geqslant 0, \mathbf{Y} \neq 0).$$

If G is a connected graph then all the eigencomponents corresponding to the largest eigenvalue of $A(G)$ are positive.

Lemma 2.2 ([23]). *If B is a symmetric $n \times n$ matrix with eigenvalues $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_n$ then for any $\mathbf{X} \in R^n(\mathbf{X} \neq 0)$,*

$$\mathbf{X}^T B\mathbf{X} \leqslant \rho_1 \mathbf{X}^T\mathbf{X}. \tag{2}$$

Equality holds if and only if $\mathbf{X}$ *is an eigenvector of B corresponding to the largest eigenvalue ρ_1.*

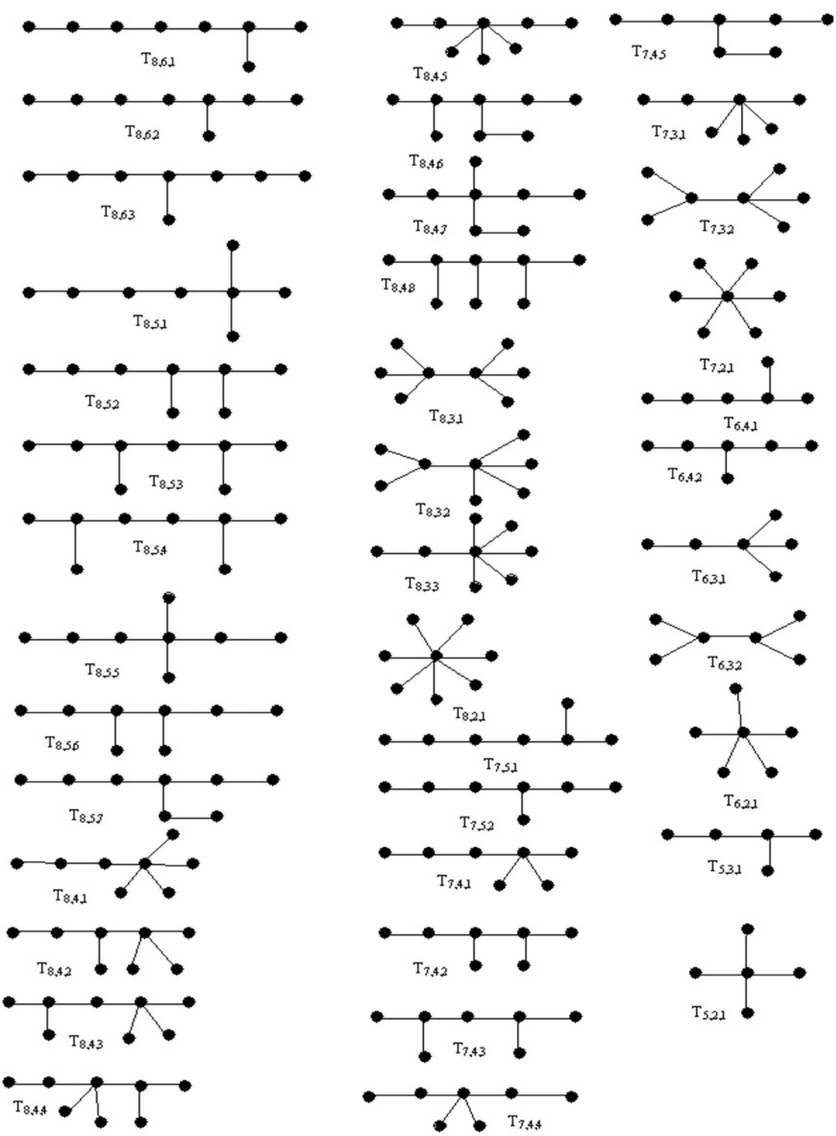

Fig. 1. Tree $T_{n,d,k}$ of order n, diameter d and $k = 1, 2, \ldots$.

Lemma 2.3 ([4]). *Let G be a connected graph of order n. Then*

$$\lambda_1 \leqslant \max_{i\sim j} \sqrt{d_i d_j},$$

where d_i is the degree of the vertex v_i in G.

Lemma 2.4 ([5]). *Let G be a connected graph of order n. Then*

$$\lambda_1 \leqslant n - 1$$

with equality holding if and only if $G \cong K_n$.

If I is an interval of the real line, denote by $m_G(I)$ the number of eigenvalues of $L(G)$, multiplicities included, that belong to I.

Lemma 2.5 ([14]). *Let G be a simple graph. Then $q(G) \leqslant m_G[0, 1)$.*

Now if an edge is added into a graph G, then none of its Laplacian eigenvalues can decrease, while the trace of the Laplacian matrix increases by 2. As So has observed in [22], if an edge is added to G and the Laplacian eigenvalues only change by integer quantities, there are just two possibilities that can arise: either one eigenvalue increases by 2, or two eigenvalues increase by 1. These scenarios are known as spectral integral variation in one place and spectral integral variation in two places, respectively.

Lemma 2.6 ([16]). *Let G be a graph on vertices labelled $v_1, v_2, \ldots, v_n$, and suppose that vertices v_1 and v_2 of G are not adjacent. Form $\widehat{G}$ from G by adding the edge between vertices v_1 and v_2. Then spectral integral variation occurs in one place if and only if in G, vertices v_1 and v_2 have the same set of neighbors. In the case that spectral integral variation occurs in one place, the eigenvalue of G that increases by 2 is given by the degree of vertex v_1 (equivalently, the degree of vertex v_2).*

3. Conjectures on index and algebraic connectivity of graphs

A path with n vertices is denoted by P_n. The near path Q_n is the tree on n vertices obtained from a path $P_{n-1} : v_1 v_2 \cdots v_{n-2} v_{n-1}$ by attaching a new pendant edge $v_{n-2}v_n$ at v_{n-2}. Let $Q'_n = Q_n + v_{n-1}v_n$, that is, Q'_n is a graph obtained from Q_n by adding an edge between vertices v_{n-1} and v_n in Q_n. So we have $Q'_n = Ki_{n,3}$.

Lemma 3.1 ([21], Theorem 3.3). *Let T be a tree of order $n \geqslant 9$ and $T \neq P_n, Q_n$. Then we have*

$$a(Q_n) < a(T).$$

Now we are ready to give a proof of Conjecture 1.

Theorem 3.2. *Let G be a connected graph on $n (n > 3)$ vertices with girth $g (g \geqslant 3)$ and algebraic connectivity a. Then $a + g$ and $a \cdot g$ are minimum for the kite $Ki_{n,3}$.*

Proof. For $n = 4$, one can see easily that $a + g$ and $a \cdot g$ are minimum for the kite $Ki_{4,3}$. Otherwise, $n \geqslant 5$. Let G be a connected graph with girth $g (g \geqslant 3)$ and let T be its spanning tree, different from P_n and Q_n. Since T is a spanning tree of G, we have

$$a(T) \leqslant a(G). \tag{3}$$

If $n \geqslant 9$, then by Lemma 3.1,

$$a(Q_n) < a(T). \tag{4}$$

For $5 \leqslant n \leqslant 8$, all the trees, not isomorphic to P_n, are in Fig. 1 in which we obtain the algebraic connectivity for each tree by Mathematica [18].

$a(T_{8,6,1}) = 0.167, a(T_{8,6,2}) = 0.186, a(T_{8,6,3}) = 0.198, a(T_{8,5,1}) = 0.202, a(T_{8,5,2}) = 0.224, a(T_{8,5,3}) = 0.214$, $a(T_{8,5,4}) = 0.186$, $a(T_{8,5,5}) = 0.254$, $a(T_{8,5,6}) = 0.251$, $a(T_{8,5,7}) = 0.243$, $a(T_{8,4,1}) = 0.277$, $a(T_{8,4,2}) = 0.289$, $a(T_{8,4,3}) = 0.238$, $a(T_{8,4,4}) = 0.319$, $a(T_{8,4,5}) = 0.382$, $a(T_{8,4,6}) = 0.306$, $a(T_{8,4,7}) = 0.382$, $a(T_{8,4,8}) = 0.268$, $a(T_{8,3,1}) = 0.354$, $a(T_{8,3,2}) = 0.374$, $a(T_{8,3,3}) = 0.452$.

$a(T_{7,5,1}) = 0.225, a(T_{7,5,2}) = 0.26, a(T_{7,4,1}) = 0.296, a(T_{7,4,2}) = 0.322, a(T_{7,4,3}) = 0.268, a(T_{7,4,4}) = 0.382$, $a(T_{7,4,5}) = 0.382$, $a(T_{7,3,1}) = 0.466$, $a(T_{7,3,2}) = 0.398$.

$a(T_{6,4,1}) = 0.325$, $a(T_{6,4,2}) = 0.382$, $a(T_{6,3,1}) = 0.486$, $a(T_{6,3,2}) = 0.438$, $a(T_{5,3,1}) = 0.519$, $a(T_{n,2,1}) = 1.0, n = 4, 5, 6, 7, 8$.

Thus (4) holds for $5 \leqslant n \leqslant 8$. From above result and (3), we get

$$a(Q_n) < a(G).$$

By Lemma 2.5, we have $m_{Q_n}[0, 1) \geqslant 2$ for $n \geqslant 5$, that is, $a(Q_n) < 1$. Again by Lemma 2.6, we conclude that an eigenvalue 1 of Q_n increases by 2 in Q_n' and the Laplacian spectrum of Q_n overlaps the Laplacian spectrum of Q_n' in the remaining $n-1$ places. So we have $a(Q_n') = a(Q_n)$, that is, $a(Ki_{n,3}) = a(Q_n') < a(G)$.

Now, let G be a connected graph with girth $g (g \geqslant 3)$, whose only spanning trees are P_n and Q_n. If G is not isomorphic to kite $Ki_{n,3}$, then there exists an spanning tree T^* of G such that T^* is neither P_n nor Q_n (for example, $G = G_1$ and $T^* = T_1$ in Fig. 2), a contradiction. Thus $G \cong Ki_{n,3}$.

Hence $a + g$ and $a \cdot g$ are minimum for kite $Ki_{n,3}$. □

Now we obtain the spectral radius of kite $Ki_{n,n-1}$ in the following Lemma 3.3.

Lemma 3.3. *Let $Ki_{n,n-1}$ be a kite of order n. Then the spectral radius of kite $Ki_{n,n-1}$ is $\lambda_1 = n - 2 + t$, where $t (0 < t < 1)$ is given by*

$$t^3 + (2n-3)t^2 + (n^2 - 3n + 1)t - 1 = 0. \tag{5}$$

Proof. First we have to show that

$$\lambda_1(Ki_{n,n-1}) > n - 2. \tag{6}$$

Let v_n be the pendant vertex in $Ki_{n,n-1}$. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be any non-zero vector in R^n. Then $\mathbf{X}^T\mathbf{X} = \sum_{i=1}^n x_i^2$ and $\mathbf{X}^T A(G)\mathbf{X} = 2\sum_{i\sim j} x_i x_j$. By Lemma 2.2, we have

$$\lambda_1(G) \geqslant \frac{2\sum_{i\sim j} x_i x_j}{\sum_{i=1}^n x_i^2}. \tag{7}$$

Putting $G = Ki_{n,n-1}$ and $\mathbf{X} = \left(\underbrace{\frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n-1}}, \cdots, \frac{1}{\sqrt{n-1}}}_{n-1}, 0\right)$ in (7), we get that $\lambda_1 \geqslant n-2$.

We can see easily that $\mathbf{X} = \left(\underbrace{\frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n-1}}, \cdots, \frac{1}{\sqrt{n-1}}}_{n-1}, 0\right)$ is not an eigenvector corresponding eigenvalue λ_1 for $Ki_{n,n-1}$. Thus we get (6). By Lemma 2.4, we have that

$$n - 2 < \lambda_1 < n - 1.$$

From this we can write $\lambda_1 = n - 2 + t$, where $0 < t < 1$.

Let x_1 and x_2 be the eigencomponents corresponding to pendant vertex of degree 1 and maximum degree vertex of degree $n-1$ of eigenvalue λ_1 of $Ki_{n,n-1}$, respectively. All the eigencomponents corresponding to the vertices of degree $n-2$ of eigenvalue λ_1 of $Ki_{n,n-1}$ are equal, x_3 say. Therefore the eigenvalue λ_1 satisfies the following system of equations:

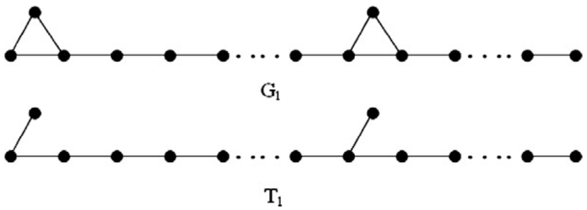

Fig. 2. Graph G_1 and Tree T_1.

$$\begin{aligned} &\lambda_1 x_1 = x_2, \\ &\lambda_1 x_2 = x_1 + (n-2)x_3 \\ \text{and } &\lambda_1 x_3 = x_2 + (n-3)x_3. \end{aligned} \tag{8}$$

From these equations, we get

$$\begin{aligned} &(\lambda_1^2 - 1)x_1 = (n-2)x_3 \\ \text{and } &(\lambda_1 - n + 3)x_3 = \lambda_1 x_1. \end{aligned}$$

From these two equations, we get

$$\lambda_1^3 - (n-3)\lambda_1^2 - (n-1)\lambda_1 + n - 3 = 0 \quad \text{as} \quad x_1, x_3 \neq 0 \text{ by Lemma 2.1}.$$

Since $\lambda_1 = n - 2 + t$, $0 < t < 1$, we get the required result in (5). □

Denote by Ki^*, a connected graph of order n obtained by deleting an edge from kite $Ki_{n,n-1}$ such that maximum degree is $n-1$. Also denote by Ki^{**}, a connected graph of order n obtained by deleting an edge from kite $Ki_{n,n-1}$ such that maximum degree is $n-2$.

Lemma 3.4. *Let H be a connected graph of order $n \geqslant 4$ obtained from $Ki_{n,n-1}$ by deleting an edge. Then the spectral radius of H is strictly less than $n-2$.*

Proof. Since H is connected, either $H = Ki^*$ or $H = Ki^{**}$. When $H = Ki^*$, the spectral radius of H is given by the following equation

$$f(\lambda) = \lambda^4 - (n-5)\lambda^3 - (3n-9)\lambda^2 - (n-1)\lambda + 2n - 8 = 0.$$

We have $f(-2) = 2 > 0$, $f\left(-2 + \frac{2}{n}\right) = -\frac{2}{n^4}(n^3 - 2n^2 + 12n - 8) < 0$, $f(0) = 2n - 8 \geqslant 0$, $f(1) = -(3n-8) < 0$, $f(n-3) = -(n^3 - 8n^2 + 21n - 16) < 0$ and $f(n-2) = 2n^2 - 7n + 2 > 0$ as $n \geqslant 4$. Thus $n - 3 < \lambda_1(H) < n - 2$.

When $H = Ki^{**}$, the spectral radius of H is given by the following equation

$$g(\lambda) = \lambda^4 - (n-4)\lambda^3 - (2n-5)\lambda^2 + (n-4)\lambda + n - 3 = 0.$$

We have $g(-3) = 7n + 27 > 0$, $g(-1) = -n + 3 < 0$, $g(0) = n - 3 > 0$, $g(1) = -(n-3) < 0$, $g(n-3) = -(n-3)^3 < 0$ and $g(n-2) = 2n^2 - 9n + 9 > 0$ as $n \geqslant 4$. Thus $n - 3 < \lambda_1(H) < n - 2$. Hence the result. □

Now we are ready to give a proof of Conjecture 2.

Theorem 3.5. *Let G be a connected graph on $n \geqslant 3$ vertices with index λ_1, vertex connectivity ν and edge connectivity κ. Then*

$$\frac{\lambda_1}{s} \leqslant n - 2 + t, \tag{9}$$

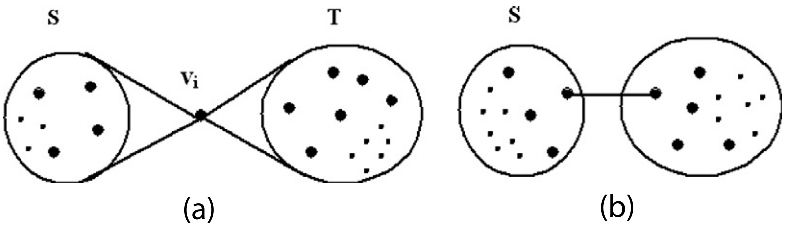

Fig. 3. (a) $\nu = 1$, (b) $\kappa = 1$.

where $s = \nu$ or κ; t is such that $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t - 1 = 0$. Moreover, the equality holds in (9) if and only if $G \cong Ki_{n,n-1}$.

Proof. For $G = P_3$, $\lambda_1 = \sqrt{2} = 1 + t$, where $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t - 1 = 0$. The equality holds in (9). For $G = K_3$, $\frac{\lambda_1}{s} = 1 < 1 + t$, the inequality (9) is strict. Now we have $n \geqslant 4$. For connected graph G, $s \geqslant 1$, where $s = \nu$ or κ. If $s \geqslant 2$, then one can see easily that the inequality (9) is strict as $\lambda_1 \leqslant n-1$ and $n \geqslant 4$. Otherwise, $s = 1$. So we have to prove

$$\lambda_1 \leqslant n - 2 + t,$$

where $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t - 1 = 0$. If $\Delta \leqslant n-2$, then $\lambda_1 \leqslant \Delta \leqslant n - 2 < n - 2 + t$. Otherwise, $\Delta = n - 1$. We consider two cases (i) $\nu = 1$, (ii) $\kappa = 1$.

Case (i): $\nu = 1$. In this case maximum degree vertex v_i of degree $n-1$ are adjacent to two vertex sets S and T in Fig. 3 (a) such that $V(G) = S \cup T \cup \{v_i\}$ and $S \cap T = \Phi$. First we assume that $|S| \geqslant 2$ and $|T| \geqslant 2$. By Lemma 2.3, we have

$$\lambda_1 \leqslant \max_{i \sim j} \sqrt{d_i d_j} \leqslant \sqrt{(n-1)(n-3)} < n - 2 + t.$$

Next we assume that either $|S| = 1$ or $|T| = 1$. Without loss of generality, we can assume that $|S| = 1$. Thus G is a super graph of star $K_{1,n-1}$ such that $\delta = 1$, that is, $K_{1,n-1} \subseteq G \subseteq K_{n,n-1}$. Hence

$$\lambda_1(G) \leqslant \lambda_1(K_{n,n-1}).$$

By Lemma 3.3 and Lemma 3.4, we conclude that

$$\lambda_1(G) \leqslant n - 2 + t$$

with equality holding if and only if $G \cong Ki_{n,n-1}$, where $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t - 1 = 0$.

Case (ii): $\kappa = 1$. In this case we separate the vertices $V(G)$ into two sets S and T such that only one edge is connected between S and T in Fig. 3(b). Since maximum degree $\Delta = n - 1$, we must have either $|S| = 1$ or $|T| = 1$. Similarly as in Case (i), we get the required result (9). Moreover, the equality holds in (9) if and only if $G \cong Ki_{n,n-1}$. $\square$

Theorem 3.6. *Let G be a connected graph on $n \geqslant 3$ vertices with index λ_1, vertex connectivity ν and edge connectivity κ. Then*

$$\lambda_1 - r \leqslant n - 3 + t, \tag{10}$$

where $r = \nu$ or κ; t is such that $0 < t < 1$ and $t^3 + (2n-3)t^2 + (n^2-3n+1)t - 1 = 0$. Moreover, the equality holds in (10) if and only if $G \cong Ki_{n,n-1}$.

Proof. First we assume that $r \geqslant 2$. For $r = \nu$ or κ,

$$\lambda_1 - r < n + t - 3 \quad \text{as} \quad \lambda_1 \leqslant n - 1 \quad \text{and} \quad 0 < t < 1.$$

Otherwise, $\nu = 1$ or $\kappa = 1$. If $\Delta \leqslant n - 2$, then

$$\lambda_1 - r \leqslant \Delta - 1 \leqslant n - 3 < n + t - 3.$$

It remains to consider the case $\Delta = n - 1$, and $\nu = 1$ or $\kappa = 1$. By Cases (i) and (ii) of Theorem 3.5, we get the required result (10). Moreover, the equality holds in (10) if and only if $G \cong Ki_{n,n-1}$. □

Denote by S_n^*, a tree of order n obtained by attaching a pendant vertex of star $K_{1,n-2}$ to a vertex of path P_2.

Lemma 3.7. *Let $S_n^*(n \geqslant 6)$ be a tree of order n. Then*

$$a(S_n^*) < 0.5.$$

Proof. By a calculation, it is not difficult to show that the characteristic polynomial of $L(S_n^*)$ is equal to

$$\det(\lambda I - L(S_n^*)) = \lambda(\lambda - 1)^{n-4}\phi(\lambda),$$

where

$$\phi(\lambda) = \lambda^3 - (n+2)\lambda^2 + (3n-2)\lambda - n = 0.$$

Now, $\phi(x) \to \infty$ as $x \to \infty$, $\phi(n-1) = -1 < 0$, $\phi(3) = -(n-3) < 0$, $\phi(2) = n - 4 > 0$, $\phi(0.5) = \frac{1}{8} + \frac{1}{4}(n-6) > 0$, $\phi(0) = -n < 0$ for $n \geqslant 6$. Thus we have $a(S_n^*) < 0.5$. □

Counter example of Conjecture 3: By Lemma 3.7, the algebraic connectivity a of $S_n^*(n \geqslant 6)$ is strictly less than 1/2. Also we have $\mu\,(S_n^*) = 2$. Thus $a \cdot \mu < 1$.

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References

[1] M. Aouchiche, Comparaison Automatisée d'Invariants en Théorie des Graphes, Ph.D. Thesis, École Polytechnique de Montréal, February 2006.
[2] M. Aouchiche, G. Caporossi, P. Hansen, Variable neighborhood search for extremal graphs. 20. Automated comparison of graph invariants, MATCH Commun. Math. Comput. Chem. 58 (2007) 365–384.
[3] M. Aouchiche, P. Hansen, A survey of automated conjectures in spectral graph theory, Linear Algebra Appl. 432 (2010) 2293–2322.
[4] A. Berman, X.-D. Zhang, On the spectral radius of graphs with cut vertices, J. Combin. Theory Ser. B 83 (2001) 233–240.
[5] L. Collatz, U. Sinogowitz, Spektren Endlicher Grafen, Abh. Math. Sem. Univ. Hamburg 21 (1957) 63–77.
[6] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, third ed., Barth, Heidelberg, 1995.
[7] K.C. Das, An improved upper bound for Laplacian graph eigenvalues, Linear Algebra Appl. 368 (2003) 269–278.
[8] K.C. Das, Some new bounds on the spectral radius of graphs, Discrete Math. 281 (2004) 149–161.
[9] K.C. Das, The Laplacian spectrum of a graph, Comput. Math. Appl. 48 (2004) 715–724.
[10] K.C. Das, S.-G. Lee, G.-S. Cheon, On the conjecture for certain Laplacian integral spectrum of graphs, J. Graph Theory 63 (2010) 106–113.
[11] S.M. Fallat, S.J. Kirkland, J.J. Molitierno, M. Neumann, On graphs whose Laplacian matrices have distinct integer eigenvalues, J. Graph Theory 50 (2005) 162–174.
[12] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973) 298–305.
[13] F.R. Gantmacher, Chelsea Publishing Company, New York, NY, 1974.
[14] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218–238.
[15] R. Grone, R. Merris, The Laplacian spectrum of a graph II*, SIAM J. Discrete Math. 7 (2) (1994) 221–229.
[16] S.J. Kirkland, Completion of Laplacian integral graphs via edge addition, Discrete Math. 295 (2005) 75–90.
[17] J.-S. Li, D. Zhang, On Laplacian eigenvalues of a graph, Linear Algebra Appl. 285 (1998) 305–307.
[18] Wolfram Research, Inc., Mathematica, Version 7.0, Champaign, IL, 2008.
[19] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197,198 (1994) 143–176.
[20] B. Mohar, The Laplacian spectrum of graphs, Graph Theory, Combin. Appl. 2 (1991) 871–898.
[21] J.-Y. Shao, J.-M. Guo, H.-Y. Shan, The ordering of trees and connected graphs by algebraic connectivity, Linear Algebra Appl. 428 (7) (2008) 1421–1438.
[22] W. So, Rank one perturbation and its application to the Laplacian spectrum of a graph, Linear and Multilinear Algebra 46 (1999) 193–198.
[23] F. Zhang, Matrix Theory: Basic Results and Techniques, Springer-Verlag, New York, 1999.